Hodge Theory Lecture 6

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A Weakly Closed Set

We start with a definition:

Definition 0.1. The set of forms weakly cohomologous to $\omega \in \Omega^k(M)$ is

$$L_\omega = \{\omega_0 \in L^2\Omega^k(M) : \exists \alpha \in L^2\Omega^{k-1}(M) \text{ such that } \langle \omega - \omega_0, \beta \rangle_{L^2} = \langle \alpha, d^*\beta \rangle_{L^2}, \forall \beta \in \Omega^k(M)\}$$

The idea behind this is that if α, ω_0 are in fact smooth, then ω_0 is cohomologous to ω, so this is a suitable "weak" form of representing the same cohomology class.

The reason why we need such a thing is that, like smooth functions, we need a weakly closed set in some Hilbert spaces, which smooth representatives of a cohomology class are not.

Regularity Part I: Analysis

All inner products, unless otherwise specified, are L^2.

For this part read Evans' proof of regularity if you haven't already, and convince yourself it holds for systems, and thus of operators between vector bundles.

There is, however, a problem with this. He assumes already some regularity, in the form of weak derivatives, while we won't know they exist a-priori.

Therefore we need to start off differently:

Theorem 0.1. *Suppose that* $\alpha \in L^2\Omega^k(M)$ *is such that* $\langle \alpha, d\beta \rangle = \langle \alpha, d^*\gamma \rangle = 0$ *for any* $\beta \in \Omega^{k-1}(M), \gamma \in \Omega^{k+1}(M)$*. Then* $\alpha \in \Omega^k(M)$ *and* $\triangle_k \alpha = 0$

Proof. By Evans, we only need to show $\alpha \in W^{1,2}\Omega^k(M)$, since then α is a weak solution of the elliptic system $\triangle_k \alpha = 0$. By this I mean that

$$\langle d\alpha, d\beta \rangle + \langle d^*\alpha, d^*\beta \rangle = \langle (d + d^*)\alpha, (d + d^*)\beta \rangle = 0, \forall \beta \in \Omega^k(M)$$

where we take the second inner product to be canonical one on $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$.

Consider a mollifier F_ε. Then

$$\langle dF_\varepsilon \alpha, \beta \rangle = \langle \alpha, F_\varepsilon d^*\beta \rangle$$

Then

$$F_\varepsilon d^* = d^*F_\varepsilon + (F_\varepsilon d^* - d^*F_\varepsilon) = d^*F_\varepsilon + [F_\varepsilon, d^*]$$

so

$$\langle \alpha, F_\varepsilon d^*\beta \rangle = \langle \alpha, d^*F_\varepsilon \beta \rangle + \langle \alpha, [F_\varepsilon, d^*]\beta \rangle = \langle \alpha, [F_\varepsilon, d^*]\beta \rangle$$

and thus

$$\|dF_\varepsilon \alpha\|^2 = \sup_{\|\beta\|=1} \langle dF_\varepsilon \alpha, \beta \rangle \leq const \sup_{\|\beta\|=1} \|\beta\| = const$$

similarly

$$\|d^*F_\varepsilon \alpha\|^2 \leq const$$

and by the elliptic inequality at the end of lecture 3

$$\|F_\varepsilon \alpha\|_{H^1} \leq const$$

Finally, we conclude using Rellich-Kondrachov that $\alpha \in W^{1,2}\Omega^k(M)$. □

Exercise 0.1. Fill in the last 2 parts: the elliptic inequality, and Rellich-Kondrachov.(Hint: A Hilbert space is weakly closed in itself)

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Exercise 0.2. Suppose that $\alpha \in L^2\Omega^k(M)$ is such that $\langle \alpha, d\beta\rangle = \langle x, \beta\rangle$ $\langle \alpha, d^*\gamma\rangle = \langle y, \gamma\rangle$for any $\beta \in \Omega^{k-1}(M), \gamma \in \Omega^{k+1}(M)$, and for some x, y smooth forms of the appropriate degree. Then $\alpha \in \Omega^k(M)$. (Hint: Once you've shown $\alpha \in W^{1,2}\Omega^k(M)$ then it is a weak solution of an elliptic system. Which one?)

Regularity Part II: Topology

We use two theorems without proof:

Theorem 0.2. *Suppose that $S \subseteq M$ is compact, of dimension k, and without boundary. Then there is some closed $n-k$ form η_S such that*

$$\int_S \omega = \int_M \omega \wedge \eta_S$$

And

Theorem 0.3. *Suppose that ω is a smooth k-form such that*

$$\int_S \omega = 0$$

for any $S \subseteq M$ compact submanifold of dimension k, and without boundary. Then ω is exact.

Existence for Minimizers

We have

Lemma 0.1. *Suppose that* $f(\alpha) = \|\alpha\|_{L^2}$ *and* $d\omega = 0$*. Then* f *has a minimum* ω_0 *on* L_ω*.*

Proof. We have that L_ω is weakly closed, f is coercive and weakly lower-semicontinuous □

Let's looks that this minimizer. First of all, we have

Lemma 0.2. ω_0 *is smooth*

Proof. First of all, note that $\forall \beta \in \Omega^{k+1}(M)$ it holds that

$$\langle \omega_0, d^*\beta \rangle_{L^2} = \langle d\omega_0, \beta \rangle_{L^2} = \langle d(\omega - \omega_0), \beta \rangle_{L^2} = \langle \omega - \omega_0, d^*\beta \rangle_{L^2} = \left\langle \alpha, (d^*)^2\beta \right\rangle_{L^2} = 0$$

Also, note that $\forall \beta \in \Omega^{k-1}(M), \alpha \in \Omega^k(M), t \in \mathbb{R}$ it holds that

$$\omega_0 + td\beta \in L_\omega$$

and hence

$$0 = 2 \langle \omega_0, d\beta \rangle_{L^2}$$

thus it is smooth. □

Exercise 0.3. Prove the last part by expanding $\|\omega_0 + td\beta\|_{L^2}^2$ and using the minimizing property.

Finally, we conclude with

Lemma 0.3. ω, ω_0 *are cohomologous*

Proof. Combine the properties of the Hodge star(See Lee Exercises 16.18−16.22) with the topological regularity theorems □

Lemma 0.4. ω_0 *is unique*

Proof. We note that if $\omega_1 \in L_\omega$ is also harmonic then $\exists \beta_1, \beta_2 \in \Omega^{k-1}(M)$ such that $\forall \alpha \in \Omega^k(M)$

$$\langle \omega_1 - \omega_0, \alpha \rangle_{L^2} = \langle \omega_1 - \omega, \alpha \rangle_{L^2} - \langle \omega - \omega_0, \alpha \rangle_{L^2} = \langle \beta_1, d^* \alpha \rangle_{L^2} + \langle \beta_2, d^* \alpha \rangle_{L^2}$$

$$= \langle \beta_1 + \beta_2, d^* \alpha \rangle_{L^2}$$

In particular, for $\alpha = \omega_1 - \omega_0$ we find that

$$\|\omega_1 - \omega_0\|_{L^2}^2 = 0$$

□

Thus we have

Theorem 0.4. *In each de Rham cohomology class there exists a unique harmonic form*